University of California, Berkeley
Physics H7C Fall 2002 (*Strovink*)

PROBLEM SET 1

1. (Taylor and Wheeler problems 5 and 17b)
According to a scientist in the laboratory, "event G occurred before event H". To what extent might this be true universally?
(a.)
Prove that the *temporal order* of two events in the laboratory is the same as in all other inertial (Lorentz) frames if and only if the two events have either a *timelike* or a *lightlike* separation.
(b.)
If two events G and H have a timelike separation, show that a Lorentz frame can be found in which the two events occur at the *same place*. In that frame, show that the time between the two events is equal to the *proper time interval* that separates them.

2. Inertial reference frames $\mathcal{S}'$ and $\mathcal{S}$ coincide at $t' = t = 0$. Frame $\mathcal{S}'$ moves with velocity $\beta c\,\hat{x}$ with respect to $\mathcal{S}$. Leaving aside the y and z coordinates, which are the same in both systems, the Lorentz transformation between $\mathcal{S}$ and $\mathcal{S}'$ is given by

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \end{pmatrix} ,$$

where Λ is a 2×2 matrix.
(a.)
Show that the Lorentz transformation found in BFG (Eqs. (2-26) and (2-27)) is equivalent to writing

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} ,$$

where

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} .$$

(b.)
Using the properties of the hyperbolic functions, show that the expression for Λ in part (a.) is equivalent to

$$\Lambda = \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} ,$$

where η, the *boost*, is

$$\eta \equiv \tanh^{-1}\beta .$$

(c.) The *inverse* Lorentz transformation is given by

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} ct' \\ x' \end{pmatrix} ,$$

where

$$\Lambda^{-1}(\beta) = \Lambda(-\beta) .$$

Show that a Lorentz transformation followed by its inverse restores the original ct and x, *i.e.*

$$\Lambda^{-1}\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \mathcal{I} ,$$

where $\mathcal{I}$ is the unit transformation.

3. BFG problem 2.18.

4. (Taylor and Wheeler problem 51)
The clock paradox, version 3.
Can one go to a point 7000 light years away – *and return* – without aging more than 40 years? "Yes" is the conclusion reached by an engineer on the staff of a large aviation firm in a recent report. In his analysis the traveler experiences a constant "1-g" acceleration (or deceleration, depending on the stage reached in her journey). Assuming this limitation, is the engineer right in his conclusion? (For simplicity, limit attention to the first phase of the motion, during which the astronaut accelerates for 10 years – then double the distance covered in that time to find how far it is to the most remote point reached in the course of the journey.)
(a.)
The acceleration is *not* $g = 9.8$ meters per second per second relative to the laboratory frame. If it were, how many times faster than light would the spaceship be moving at the end of ten years (1 year = 31.6×10^6 seconds)? *If the acceleration is not specified with respect to the*

laboratory, then with respect to what is it specified? Discussion: Look at the bathroom scales on which one is standing! The rocket jet is always turned up to the point where these scales read one's *correct* weight. Under these conditions one is being accelerated at 9.8 meters per second per second with respect to a spaceship that (1) instantaneously happens to be riding alongside with identical velocity, but (2) is *not* being accelerated, and, therefore (3) *provides the* (momentary) *inertial frame of reference relative to which the acceleration is* g.

(b.)

How much velocity does the spaceship have after a given time? This is the moment to object to the question and to rephrase it. *Velocity* βc is not the simple quantity to analyze. The simple quantity is the *boost parameter* η. This parameter is simple because it is *additive* in this sense: Let the boost parameter of the spaceship with respect to the imaginary instantaneously comoving inertial frame change from 0 to $d\eta$ in an astronaut time $d\tau$. Then the boost parameter of the spaceship with respect to the *laboratory* frame changes in the same astronaut time from its initial value η to the subsequent value $\eta + d\eta$. Now relate $d\eta$ to the acceleration g in the instantaneously comoving inertial frame. In this frame $g\,d\tau = c\,d\beta = c\,d(\tanh\eta) = c\,\tanh(d\eta) \approx c\,d\eta$ so that

$$c\,d\eta = g\,d\tau$$

Each lapse of time $d\tau$ on the astronaut's watch is accompanied by an additional increase $d\eta = \frac{g}{c}d\tau$ in the boost parameter of the spaceship. In the laboratory frame the total boost parameter of the spaceship is simply the sum of these additional increases in the boost parameter. Assume that the spaceship starts from rest. Then its boost parameter will increase linearly with *astronaut* time according to the equation

$$c\eta = g\tau$$

This expression gives the boost parameter η of the spaceship in the *laboratory* frame at any time τ in the *astronaut's* frame.

(c.)

What laboratory distance x does the spaceship cover in a given astronaut time τ? At any instant the velocity of the spaceship in the laboratory frame is related to its boost parameter by the equation $dx/dt = c\tanh\eta$ so that the distance dx covered in *laboratory* time dt is

$$dx = c\,\tanh\eta\,dt$$

Remember that the time between ticks of the astronaut's watch $d\tau$ appear to have the larger value dt in the laboratory frame (time dilation) given by the expression

$$dt = \cosh\eta\,d\tau$$

Hence the laboratory distance dx covered in *astronaut* time $d\tau$ is

$$dx = c\,\tanh\eta\,\cosh\eta\,d\tau = c\,\sinh\eta\,d\tau$$

Use the expression $c\eta = g\tau$ from part b to obtain

$$dx = c\,\sinh\left(\frac{g\tau}{c}\right)d\tau$$

Sum (integrate) all these small displacements dx from zero astronaut time to a final astronaut time to find

$$x = \frac{c^2}{g}\left[\cosh\left(\frac{g\tau}{c}\right) - 1\right]$$

This expression gives the laboratory *distance* x covered by the spaceship at any time τ in the astronaut's frame.

(d.)

Plugging in the appropriate numerical values, determine whether the engineer is correct in his conclusion reported at the beginning of this exercise.

5. Starting from the *relativistic addition theorem for velocities*, BFG Eq. (2-41), prove the *addition formula for hyperbolic tangents*:

$$\tanh(a-b) = \frac{\tanh a - \tanh b}{1 - \tanh a \tanh b}\,.$$

6. The now retired Bevatron at Berkeley Lab is famous for having produced the first observed antiprotons (you may have glimpsed white-maned Nobelist Owen Chamberlain, one of the first observers in 1958, being helped to his seat at Physics Department colloquia). An economical reaction for producing antiprotons is

$$p + p \rightarrow p + p + p + \bar{p} \; ,$$

where the first proton is part of a beam, the second is at rest in a target, and $\bar{p}$ is an antiproton. Because of the CPT theorem, both p and $\bar{p}$ must have the same mass ($mc^2 = 0.94 \times 10^9$ eV).

At threshold, all four final state particles have essentially zero velocity *with respect to each other*. What is the beam energy in that case? (The actual Bevatron beam energy was 6×10^9 eV).

7. "Surface" muon beams are important tools for investigating the properties of condensed matter samples as well as fundamental particles. Protons from a cyclotron produce π^+ mesons (quark-antiquark pairs) that come to rest near the surface of a solid target. The pion then decays isotropically to an (anti)muon (μ^+) and a neutrino (ν) via

$$\pi^+ \rightarrow \mu^+ + \nu \; .$$

Some of the muons can be captured by a beam channel and transported in vacuum to an experiment. In the limit that the mother pion decays at the surface of the target (so that the daughter muon traverses negligible material), the beam muons have uniform speed. For the purposes of this problem, consider a muon to have $\frac{3}{4}$ of the rest mass of a pion; neglect the neutrino mass. Show that the surface muons travel at a speed which is a fraction $\beta_0 = 0.28$ of the speed of light.

8. As described in the TESLA Design Report written in 2001 at the DESY laboratory in Hamburg, Germany, electrons of rest mass $mc^2 = 0.5 \times 10^6$ eV would be accelerated to a total energy of 0.25×10^{12} eV over an active length of 10.9 km. (Positrons (anti-electrons), after similar acceleration, would collide head-on with the electrons. Individual electrons and positrons would mutually annihilate, releasing 0.5×10^{12} eV of energy in a pure form that existed just after the Big Bang.)

(a.)
To what boost η are the electrons ultimately brought?

(b.)
Assuming that the electrons are subjected to a uniform acceleration as observed in their comoving inertial frame, how many g's of acceleration do they feel?

(c.)
As observed at the DESY lab, for what time interval is each electron in flight? What is the corresponding *proper* time interval? Evaluate the ratio of the two intervals (a sort of average γ factor).